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Analysis of a Model of a Cocurrent Packed-Bed Column with Periodic Inlet Conditions

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Abstract—We consider a model of a process in a cocurrent packed-bed column with the simplest kinetics of the rate of liquid holdup formation. Under study is the dependence of it on the time-periodic velocities of the liquid and gas flows in a neighborhood of the stationary values of these velocities. The principal terms of the asymptotics in time are found for the liquid holdup and the flow velocities. We detect the growth of the amplitudes of the sinusoidal oscillations of the liquid holdup and the liquid and gas velocities with respect to the column length. The constructed model is used for design of the cocurrent two-phase columns.

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1. STATEMENT OF THE PROBLEM

The one-dimensional transport of a two-phase flow of liquid and gas is described by the balance equations as follows [1, 2]:

$$\frac{\partial h}{\partial t} - \frac{\partial v_L}{\partial z} = 0, \qquad \frac{\partial h}{\partial t} + \frac{\partial v_G}{\partial z} = 0, \qquad \frac{\partial h}{\partial t} = -k(h - h^\infty(v_L, v_G)). \tag{1.1}$$

The boundary conditions for the gas velocity v_G and the liquid velocity v_L are given for $z = 0$ at the inlet to the column:

$$v_G|_{z=0} = \psi(t), \qquad v_L|_{z=0} = \varphi(t), \tag{1.2}$$

and the initial data for v_G and v_L and the function $h(z,t)$ have the form

$$v_G|_{t=0} = v_G^0(z), \qquad v_L|_{t=0} = v_L^0(z), \qquad (h(z,t) - h^\infty(v_L, v_G)|_{t=0} = 0. \tag{1.3}$$

Here $h(z,t)$ stands for the concentration of the liquid at a point (z,t) and $h^\infty(v_L, v_G)$ is a sufficiently smooth function of (v_L, v_G).

Integrating the first two equations of (1.1), we express v_L and v_G as

$$v_L(z,t) = \int_0^z \frac{\partial h}{\partial t}\, dz + \varphi(t), \tag{1.4}$$

$$v_G(z,t) = -\int_0^z \frac{\partial h}{\partial t}\, dz + \psi(t). \tag{1.5}$$

The insertion of v_L and v_G from (1.4) and (1.5) in the last equation in (1.1) gives a nonlinear integro-differential equation for finding $h(z,t)$. In general, a solution to this equation for an arbitrary nonlinear function $h^\infty(v_L, v_G)$ and initial data from (1.3) does not exist "in the large" for all $t > 0$. We can choose

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a nonlinear function $h^\infty(v_G, v_L)$ such that the solution $h(z,t)$ will destroy in finite time. But it is also possible to prove the well-posedness of problem (1.1)–(1.3) "in the small" with respect to time in the class $C^1(\overline{Q}_T)$, where $\overline{Q}_T = \{(z,t) : 0 \le z \le L,\ 0 \le t \le T\}$, for smooth initial and boundary data satisfying the compatibility conditions up to the first order and sufficiently small $T > 0$.

In mathematical modeling of chemical processes, it is necessary to prove not only the well-posedness of the problem, but also to carry out a qualitative analysis of solutions to the corresponding equations. This study allows us (for example, see [3, 4]) to initially confirm the coincidence of the properties of the models with the available experimental data at the qualitative level as well as to predict the possibility of new effects and their use in the control and optimization of processes. Moreover, it is first necessary to find the conditions (parameters) which would guarantee the stability of the stationary states and the stabilization of nonstationary regimes to them from different initial conditions.

In our case, the experimental data obtained in the cocurrent column show that the concentrations of the liquid and the gas as well as their velocities v_L and v_G stabilize over time to constant values if the values v_L and v_G at the inlet $z = 0$ to the column are constant (i.e., do not depend on t).

The equations imply that if v_L and v_G are some constants then the numbers v_L, v_G, and $h^\infty(v_L, v_G)$ constitute stationary solutions to (1.1). It can be proved that, for the analysis of the stability of stationary solutions to (1.1)–(1.3), it is possible to apply the linearization principle (the Lyapunov's First Method) justified for the general parabolic problems in the works by T. I. Zelenyak, V. S. Belonosov, and M. P. Vishnevskii and, for a class of hyperbolic problems, by N. A. Eltysheva and other authors (for instance, see [5–7]).

In the case of the asymptotic stability of a stationary solution, the linearization principle guarantees the solvability of the nonlinear problem "in the large" with respect to time from the initial data taken in a certain neighborhood of this stationary solution. Moreover, the solutions to the nonlinear problem with the initial data in a small neighborhood of a stationary solution stabilizes to this stationary solution, which agrees well with the experimental data. Therefore, leaving the same notation for the deviations of solutions from the stationary solutions for the sake of simplicity, we linearize the equations on this stationary solution. Instead of the original system, we obtain a system of the previous kind (1.1), where the new function $h^\infty(v_L, v_G)$ is linear:

$$h^\infty(v_L, v_G) = \lambda v_L + \gamma v_G + \varkappa. \tag{1.6}$$

The constant parameters λ, γ, and $\varkappa$ are given by the formulas

$$\lambda = \frac{\partial h^\infty(v_L, v_G)}{\partial v_L}, \qquad \gamma = \frac{\partial h^\infty(v_L, v_G)}{\partial v_G}, \qquad \varkappa = h^\infty(v_L, v_G)$$

calculated from the original function $h^\infty(v_L, v_G)$ at the stationary solution under consideration. Preserve the same notation for the solution to the linearized problem and assume in what follows that the function $h^\infty(v_L, v_G)$ has the form (1.6), where we, without loss of generality, assume that $\varkappa = 0$. In the sequel, we suppose that the boundary data (1.2) and the initial conditions (1.3) are some differentiable functions of their variables and satisfy the compatibility conditions up to the first order.

Multiplying (1.4) by λ and (1.5) by γ, summing up and using (1.6), we obtain

$$\frac{\partial h}{\partial t} = -k(h - F(t)) + m \int_0^z \frac{\partial h}{\partial t}\, dz, \tag{1.7}$$

where $F(t) = \lambda\varphi(t) + \gamma\psi(t)$ and $m = k(\lambda - \gamma)$. The initial data in (1.3) imply

$$(h(z,t) - F(t))|_{t=0} = 0. \tag{1.8}$$

2. WELL-POSEDNESS OF THE PROBLEM

For analyzing the well-posedness of the linearized problem, it suffices to prove the well-posedness of problem (1.7), (1.8) since, by (1.4) and (1.5), the functions $v_L(z,t)$ and $v_G(z,t)$ are calculated explicitly via $h(x,t)$.

Rewrite (1.7), (1.8) as the integro-differential equation

$$\frac{\partial}{\partial t}\left(h - m\int_0^z h(x,t)\,dx\right) = -kh + kF(t), \qquad h(z,0) = F(0) = c. \tag{2.1}$$

Introduce the function

$$q(z,t) = h - m\int_0^z h(x,t)\,dx. \tag{2.2}$$

The initial condition $h(z,t)|_{t=0} = c$ yields the initial condition for $q(z,t)$:

$$q(z,0) = c(1 - mz). \tag{2.3}$$

The solution to (2.2) can be found in the form

$$h(z,t) = q(z,t) + m\int_0^z e^{m(z-x)}q(x,t)\,dx. \tag{2.4}$$

Formulas (2.2) and (2.4) give $h(z,t)$ whenever $q(z,t)$ is known or yield $q(z,t)$ whenever $h(z,t)$ is available. Therefore, it suffices to establish the solvability of the problem:

$$\frac{\partial q(z,t)}{\partial t} = -k\left(q(z,t) + m\int_0^z e^{-m(x-z)}q(x,t)\,dx\right) + kF(t), \qquad q(z,0) = c(1-mz). \tag{2.5}$$

This problem is equivalent to the equation obtained by the time integration of (2.5):

$$q(z,t) = -k\int_0^t q(z,\tau)\,d\tau - km\int_0^t\int_0^z e^{-m(x-z)}q(x,\tau)\,dx d\tau + f(z,t), \tag{2.6}$$

where $f(z,t) = c(1-mz) + k\int_0^t F(\tau)\,d\tau$ is known.

The existence and uniqueness of a solution to (2.6) can be proved in a standard manner. Moreover, below we prove that, by introducing a new sought function, (2.5) is reduced to a Goursat problem for the telegraph equation whose solution is found by constructing a Riemann function. Therefore, assuming the well-posedness of the linearized problem (1.1)–(1.3) for $h^\infty(v_L, v_G)$ of the form (1.6) to be proved, we investigate the qualitative properties of the solution.

3. BEHAVIOR OF THE SOLUTION FOR CONSTANT BOUNDARY DATA

Study the solution of (1.1)–(1.3) in the case when the boundary conditions are constant; i.e., $\psi(t) \equiv \text{const}$ and $\varphi(t) \equiv \text{const}$. Then

$$F(t) = \lambda\varphi(t) + \gamma\psi(t) = F(0) \equiv \text{const}.$$

Putting

$$u(z,t) = h(z,t) - F(0), \tag{3.1}$$

from (1.7) we obtain the problem concerned the behavior of solutions to

$$\frac{\partial u}{\partial t} = -ku + m\int_0^z \frac{\partial u}{\partial t}\,dz \tag{3.2}$$

in a neighborhood of the zero under small perturbations of the initial data:

$$u(z,0)|_{t=0} = u^0(z). \tag{3.3}$$

As was observed above, a solution to (3.2), (3.3) exists for all $t \geq 0$ for all perturbations. On the solutions to (3.2), (3.3), we construct a Lyapunov functional in the form

$$A(u) = \int_0^L \left(u(z,t) - m \int_0^z u(x,t)\,dx \right)^2 dz. \tag{3.4}$$

The calculation of the derivative of this functional yields

$$\begin{aligned}
\frac{\partial}{\partial t} A(u) &= 2 \int_0^L \left(u(z,t) - m \int_0^z u(x,t)\,dx \right) \left(\frac{\partial u}{\partial t}(z,t) - m \int_0^z \frac{\partial u}{\partial t}(x,t)\,dx \right) dz \\
&= 2 \int_0^L \left(u(z,t) - m \int_0^z u(x,t)\,dx \right) (-ku(z,t))\,dz \\
&= -2k \int_0^L u^2(z,t)\,dz + km \int_0^L \frac{d}{dz} \left(\int_0^z u(x,t)\,dx \right)^2 dz \\
&= -2k \int_0^L u^2(z,t)\,dz + km \left(\int_0^z u(x,t)\,dx \right)^2 \Bigg|_0^L = -2k \int_0^L u^2(z,t)\,dz + km \left(\int_0^L u(x,t)\,dx \right)^2;
\end{aligned}$$

i.e.,

$$\frac{\partial}{\partial t} A(u) = -2k \int_0^L u^2(z,t)\,dz + km \left(\int_0^L u(x,t)\,dx \right)^2. \tag{3.5}$$

From (3.4) and (3.5) we see that, for $k > 0$ and $m \leq 0$,

$$A(u) \geq 0, \qquad \frac{\partial}{\partial t} A(u) \leq 0. \tag{3.6}$$

The Cauchy–Bunyakovskii inequality

$$\left(\int_0^L u(z,t)\,dz \right)^2 \leq L \int_0^L u^2(x,t)\,dx$$

and (3.5) imply that if

$$-2k + k|m|L < 0, \tag{3.7}$$

then on solutions to (3.2) we have $\frac{\partial}{\partial t} A(u) \leq 0$. Inequality (3.7) also holds for fixed k and small m and L or for given $\lambda - \gamma$, L, and small k. Furthermore, (3.7) always holds for all values of the parameters satisfying $|m|L < 2$.

Corollary 1. *If $k > 0$ and $\lambda - \gamma \leq 0$ then problem* (3.2) *has no periodic solutions.*

Proof. Equality (3.6) is possible for such parameters if and only if $u(z,t) \equiv 0$. This means that there is no nontrivial (nonzero) solution to (3.2) periodic in time. ☐

Corollary 2. *If $k > 0$, $\lambda - \gamma \le 0$, and the boundary data are constant (i.e., $\psi(t) \equiv \mathrm{const}$ and $\varphi(t) \equiv \mathrm{const}$) then problem (1.1)–(1.3) has only a constant solution as a stationary solution and has no solutions periodic in time.*

We consider the proof of the stability of this stationary solution in Section 6.

4. SOLVING THE PROBLEM WITH THE USE OF THE RIEMANN FUNCTION FOR CONSTANT INLET DATA

Solve the problem (3.2), (3.3) explicitly. Let us construct the Riemann function for the telegraph equation that follows from the previous arguments.

Rewrite (3.2), (3.3) in a convenient form:

$$\frac{\partial u}{\partial t} = -ku + m\int_0^z \frac{\partial u}{\partial t}\,dz, \qquad u(z,0) \equiv c = \mathrm{const}. \tag{4.1}$$

As in (2.2), put

$$u - m\int_0^z u(x,t)\,dx = q(z,t), \qquad q(z,0) = c(1-mz). \tag{4.2}$$

The change

$$p(z,t) = e^{kt}e^{-mz}q(z,t), \qquad p(z,0) = e^{-mz}q(z,0) \tag{4.3}$$

leads to the problem

$$\frac{\partial p}{\partial t} = -km\int_0^z p(\xi,t)\,d\xi, \qquad p(0,t) = c, \qquad p(z,0) = e^{-mz}(1-mz)c = p^0(z). \tag{4.4}$$

As in (2.2), $u(z,t)$ can be written as

$$u(z,t) = e^{-kt}e^{mz}\left[p(z,t) + m\int_0^z p(x,t)\,dx\right]. \tag{4.5}$$

Formulas (4.2)–(4.5) give us $u(z,t)$ if $p(z,t)$ is known or $p(z,t)$, if $u(z,t)$ is known.

The equation in (4.4) is reduced to the telegraph equation. Indeed, differentiating (4.4) with respect to z, we obtain

$$\frac{\partial^2 p(z,t)}{\partial z\partial t} = -kmp(z,t).$$

For problem (4.4), construct the Riemann function $R(z,t,\sigma,\tau)$ [8, p. 132] that in our case is

$$R(z,t,\sigma,\tau) = J_0\big(\sqrt{4km(z-\sigma)(t-\tau)}\,\big), \tag{4.6}$$

where $J_0(\xi)$ is the Bessel function with $J_0(0) = 1$. The properties of the Bessel function imply

$$\frac{\partial^2 R(z,t,\sigma,\tau)}{\partial z\partial t} = \frac{\partial^2 R(z,t,\sigma,\tau)}{\partial\sigma\partial\tau} = -kmR(z,t,\sigma,\tau), \tag{4.7}$$

$$R(z,t,z,\tau) = R(z,t,\sigma,t) = 1. \tag{4.8}$$

Multiplying the equation

$$\frac{\partial^2 p(\sigma,\tau)}{\partial\sigma\partial\tau} + kmp(\sigma,\tau) = 0 \tag{4.9}$$

by $R(z,t,\sigma,\tau) = J_0\big(\sqrt{4km(z-\varsigma)(t-\tau)}\big)$ and integrating by parts over $0 \le \sigma \le z$ and $0 \le \tau \le t$, and using (4.7), (4.8), we infer

$$p(z,t) = p(0,t) + \int_0^z \frac{\partial p(\sigma,0)}{\partial\sigma} R(z,t,\varsigma,0)\,d\sigma + \int_0^t p(z,\tau)\frac{\partial R(z,t,z,\tau)}{\partial\tau}\,d\tau$$

$$- \int_0^t p(0,\tau)\frac{\partial R(z,t,0,\tau)}{\partial\tau}\,d\tau = c + \int_0^z \frac{\partial p(\sigma,0)}{\partial\sigma} R(z,t,\sigma,0)\,d\sigma - c\int_0^t \frac{\partial R(z,t,0,\tau)}{\partial\tau}\,d\tau$$

$$= c + \int_0^z \frac{\partial p(\sigma,0)}{\partial\sigma} R(z,t,\sigma,0)\,d\sigma - c\big(R(z,t,0,t) - R(z,t,0,0)\big)$$

$$= cR(z,t,0,0) + \int_0^z \frac{\partial p^0(\sigma)}{\partial\sigma} R(z,t,\sigma,0)\,d\sigma = cJ_0\big(\sqrt{4kmzt}\,\big) + \int_0^z \frac{\partial p^0(\sigma)}{\partial\sigma} J_0\big(\sqrt{4kmt(z-\sigma)}\,\big)\,d\sigma.$$

Thus, we have the following for the solution to (4.4):

$$p(z,t) = cJ_0\big(\sqrt{4kmzt}\,\big) + \int_0^z \frac{\partial p^0(\sigma)}{\partial\sigma} J_0\big(\sqrt{4kmt(z-\sigma)}\,\big)\,d\sigma, \tag{4.10}$$

which we also transform to a convenient form:

$$p(z,t) = cJ_0\big(\sqrt{4kmzt}\,\big) + \int_0^z \frac{\partial p^0(\sigma)}{\partial\sigma} J_0\big(\sqrt{4kmt(z-\sigma)}\,\big)\,d\sigma$$

$$= p^0(z) - \int_0^z p^0(\sigma)\frac{\partial J_0\big(\sqrt{4kmt(z-\sigma)}\,\big)}{\partial\sigma}\,d\sigma = p^0(z) + \int_0^z p^0(\sigma)\frac{\partial J_0\big(\sqrt{4kmt(z-\sigma)}\,\big)}{\partial z}\,d\sigma$$

$$= p^0(z) + \frac{\partial}{\partial z}\int_0^z p^0(\sigma) J_0\big(\sqrt{4kmt(z-\sigma)}\,\big)\,d\sigma - p^0(z).$$

We finally have

$$p(z,t) = \frac{\partial}{\partial z}\int_0^z p^0(\sigma) J_0\big(\sqrt{4kmt(z-\sigma)}\,\big)\,d\sigma. \tag{4.11}$$

5. SOLVING THE PROBLEM WITH THE USE OF THE RIEMANN FUNCTION FOR GENERAL INLET CONDITIONS

If the boundary data are nonconstant then $F(t) = \lambda\varphi(t) + \gamma\psi(t) + \varkappa$ is not constant either. Therefore, problem (2.1) can be rewritten as

$$\frac{\partial}{\partial t}\left(h - m\int_0^z h(x,t)\,dx\right) = -kh + kF(t), \qquad h(z,t)|_{t=0} = F(0) = F_0 \equiv \text{const}. \tag{5.1}$$

For $z = 0$, the solution to (5.1) has the form

$$h(0,t) = e^{-kt}\left(F_0 + \int_0^t kF(\tau)e^{k\tau}\,d\tau\right).$$

After differentiating (5.1) with respect to z, we can see that $h(z,t)$ is a solution to the following problem:

$$\frac{\partial^2 h}{\partial z \partial t} = -k\frac{\partial h}{\partial z} + m\frac{\partial h}{\partial t}, \qquad h(0,t) = e^{-kt}\left(F_0 + \int_0^t kF(\tau)e^{k\tau}\,d\tau\right), \qquad h(z,0) = F_0. \tag{5.2}$$

Define the function $p(z,t)$ as

$$h(z,t) = p(z,t)e^{-kt+mz}. \tag{5.3}$$

This leads to the problem

$$\frac{\partial^2 p}{\partial z \partial t} = -kmp, \qquad p(0,t) = \left(F_0 + \int_0^t kF(\tau)e^{k\tau}\,d\tau\right), \qquad p(z,0) = F_0e^{-mz}. \tag{5.4}$$

It is easy to prove that if $p(z,t)$ is solution to (5.4) then $h(z,t) = p(z,t)e^{-kt+mz}$ is a solution to (5.1). Above we have shown that a solution to this problem exists and is unique. Therefore, it suffices to find the explicit form of the solution to (5.4). The use of the Riemann function in the integral representation for the solution to (5.4) gives

$$\begin{aligned} p(z,t) &= p(0,t) + \int_0^z \frac{\partial p(\sigma,0)}{\partial\sigma}R(z,t,\sigma,0)\,d\sigma - \int_0^t p(0,\tau)\frac{\partial R(z,t,0,\tau)}{\partial\tau}\,d\tau \\ &= F_0R(z,t,0,0) + \int_0^z \frac{\partial p(\sigma,0)}{\partial\sigma}R(z,t,\sigma,0)\,d\sigma + \int_0^t kF(\tau)e^{k\tau}R(z,t,0,\tau)\,d\tau \\ &= F_0J_0(\sqrt{4kmzt}\,) + \int_0^z \frac{\partial p(\sigma,0)}{\partial\sigma}J_0(\sqrt{4kmt(z-\sigma)}\,)\,d\sigma + \int_0^t kF(\tau)e^{k\tau}J_0(\sqrt{4kmz(t-\tau)}\,)\,d\tau. \end{aligned}$$

Then $h(z,t)$ can be written as the sum of three summands:

$$\begin{aligned} h(z,t) = e^{-kt+mz}\Bigg[F_0J_0(\sqrt{4kmzt}\,) - mF_0\int_0^z e^{-m\sigma}J_0(\sqrt{4kmt(z-\sigma)}\,)\,d\sigma \\ + \int_0^t kF(\tau)e^{k\tau}J_0(\sqrt{4kmz(t-\tau)}\,)\,d\tau\Bigg] = I_1(z,t) + I_2(z,t) + I_3(z,t). \end{aligned} \tag{5.5}$$

6. QUALITATIVE PROPERTIES OF SOLUTIONS TO THE PROBLEM

Using the properties of the Bessel function, we will study the qualitative behavior of solutions for the concentration and the mean concentration of the liquid in a volume. Using the properties of the Bessel function, we obtain the following estimate for positive $k > 0$ and $m \geq 0$:

$$|p(z,t)| \leq K_1\left(\max|p^0(z)| + \max\left|\frac{dp^0(z)}{dz}\right|\right),$$

which gives the following for a solution $u(z,t)$ in (3.2):

$$|u(z,t)| = |h(z,t) - F(0)| \leq e^{-kt}e^{mz}\left|p(z,t) + m\int_0^z p(x,t)\,dx\right| \leq K_2e^{-kt}. \tag{6.1}$$

This means the asymptotic stability of a stationary solution for the concentration under constant inlet velocities

$$F(t) = \lambda\varphi(t) + \gamma\psi(t) = F(0) \equiv \text{const}.$$

If the inlet data in the packed-bed column are periodic functions with period τ_0 then the concentration of the liquid tends to some function that is periodic with the same period τ_0. Prove that.

By the boundedness of the integrands and the properties of the Bessel functions, there are some constants k_1 and k_2 such that $|I_i(z,t)| \le k_i e^{-kt}$ for $i = 1, 2$, where

$$I_1(z,t) = e^{-kt+mz}[F_0 J_0(\sqrt{4kmzt}\,)],$$

$$I_2(z,t) = e^{-kt+mz}\left[-mF_0 \int\limits_0^z e^{-m\sigma} J_0(\sqrt{4kmt(z-\sigma)}\,)\,d\sigma\right].$$

Then the difference of $h(z,t)$ in (5.5) and $I_3(z,t)$, where

$$I_3(z,t) = e^{-kt+mz}\left[\int\limits_0^t kF(\tau)e^{k\tau} J_0(\sqrt{4kmz(t-\tau)}\,)\,d\tau\right], \tag{6.2}$$

satisfies the following estimate with some constant $k_0 > 0$:

$$|h(z,t) - I_3(z,t)| = |I_1 + I_2| \le k_0 e^{-kt}. \tag{6.3}$$

Using the periodicity of $F(t)$, prove that $I_3(z,t)$ stabilizes to a periodic function with the same period as $t \to \infty$. We infer

$$I_3(z,t+\tau_0) = e^{-kt+mz} \int\limits_0^{t+\tau_0} kF(\tau)e^{k(\tau-\tau_0)} J_0(\sqrt{4kmz(t+\tau_0-\tau)}\,)\,d\tau$$

$$= e^{-kt+mz} \int\limits_{-\tau_0}^t kF(\sigma+\tau_0)e^{k\sigma} J_0(\sqrt{4kmz(t-\sigma)})\,d\sigma = e^{-kt+mz} \int\limits_{-\tau_0}^t kF(\sigma)e^{k\sigma} J_0(\sqrt{4kmz(t-\sigma)}\,)\,d\sigma$$

$$= e^{-kt+mz} \int\limits_{-\tau_0}^0 kF(\sigma)e^{k\sigma} J_0(\sqrt{4kmz(t-\sigma)}\,)\,d\sigma + e^{-kt+mz} \int\limits_0^t kF(\sigma)e^{k\sigma} J_0(\sqrt{4kmz(t-\sigma)}\,)\,d\sigma$$

$$= e^{-kt+mz} \int\limits_{-\tau_0}^0 kF(\sigma)e^{k\sigma} J_0(\sqrt{4kmz(t-\sigma)}\,)\,d\sigma + I_3(z,t),$$

which implies

$$|I_3(z,t+\tau_0) - I_3(z,t)| \le K_3 e^{-kt}, \tag{6.4}$$

and the previous inequalities imply

$$|h(z,t+\tau_0) - h(z,t)| \le |h(z,t+\tau_0) - I_3(z,t+\tau_0)| + |I_3(z,t+\tau_0) - I_3(z,t)| + |I_3(z,t) - h(z,t)| \le K_4 e^{-kt}.$$

Consider the asymptotics of the integral $I_3(z,t)$ for large times by specifying the form of $F(t)$. Put

$$F(t) = c + \lambda\varphi(t) + \gamma\psi(t) = c + A\sin(\omega t),$$

where A and ω are the amplitude and frequency of the oscillations of the velocities at the inlet near stationary value $c = F(0)$. This enable us not only to calculate an explicit asymptotic formula for the solution but also allow us to explain the experimentally observed periodic waves of the liquid "running" over the column length with a considerably increasing concentration amplitude near the outlet [1].

Let us use the formula for integrals of the Bessel functions [9, p. 669]:

$$B = \int_0^\infty J_n(\lambda\rho)e^{-\theta\lambda^2}\lambda^{n+1}\,d\lambda = \frac{1}{2\theta}\left(\frac{\rho}{2\theta}\right)^n e^{-\rho^2/(4\theta)}. \tag{6.5}$$

We demonstrated above that the asymptotic of the concentration $h(z,t)$ is determined by the integral $I_3(z,t)$. Perform the change of variables $t-\tau = \lambda^2/(4kmz)$ in (6.2). By convergence as $t\to\infty$, the integral in (6.2) is representable as $I_3(z,t) = l_1(z,t) + l_2(z,t)$, where

$$l_1(z,t) = e^{mz}\int_0^\infty kF\left(t-\frac{\lambda^2}{4kmz}\right)e^{-\lambda^2/(4mz)}J_0(\lambda)\,d\left(\frac{\lambda^2}{4kmz}\right),$$

$$l_2(z,t) = -e^{mz}\int_{\sqrt{4kmzt}}^\infty kF\left(t-\frac{\lambda^2}{4kmz}\right)e^{-\lambda^2/(4mz)}J_0(\lambda)\,d\left(\frac{\lambda^2}{4kmz}\right).$$

Estimate $l_2(z,t)$ using the boundedness of the functions therein. There exists a constant K_5 for which

$$|l_2(z,t)| \le K_5\left|\int_{\sqrt{4kmzt}}^\infty e^{-\lambda^2/(4mz)}\,d\left(\frac{\lambda^2}{4kmz}\right)\right| = K_5e^{-kt}.$$

Now, calculate $l_1(z,t)$ by putting $F(t) = c + A\sin(\omega t)$ and using the Euler's formulas and (6.5):

$$\begin{aligned}
l_1(z,t) &= e^{mz}\int_0^\infty F\left(t-\frac{\lambda^2}{4kmz}\right)J_0(\lambda)\,d(e^{-\lambda^2/(4mz)})\\
&= e^{mz}\int_0^\infty \left(c + A\sin\omega\left(t-\frac{\lambda^2}{4kmz}\right)\right)J_0(\lambda)\,d(e^{-\lambda^2/(4mz)})\\
&= e^{mz}c\int_0^\infty J_0(\lambda)\,d(e^{-\lambda^2/(4mz)}) + e^{mz}A\int_0^\infty \sin\omega\left(t-\frac{\lambda^2}{4kmz}\right)J_0(\lambda)\,d(e^{-\lambda^2/(4mz)})\\
&= e^{mz}c\int_0^\infty J_0(\lambda)\frac{\lambda}{2mz}e^{-\lambda^2/(4mz)}\,d\lambda\\
&\quad + e^{mz}A\int_0^\infty \frac{\lambda}{2mz}e^{-\lambda^2/(4mz)}\frac{e^{i\omega(t-\lambda^2/(4kmz))} - e^{-i\omega(t-\lambda^2/(4kmz))}}{2i}J_0(\lambda)\,d\lambda\\
&= c + Ae^{mz}\int_0^\infty \frac{\lambda}{2mz}e^{-\lambda^2/(4mz)}\frac{e^{i\omega(t-\lambda^2/(4kmz))} - e^{-i\omega(t-\lambda^2/(4kmz))}}{2i}J_0(\lambda)\,d\lambda\\
&= c + 2A\mathrm{Re}\left(e^{mz}\int_0^\infty \frac{\lambda}{2mz}e^{-\lambda^2/(4mz)}\frac{e^{i\omega(t-\lambda^2/(4kmz))}}{2i}J_0(\lambda)\,d\lambda\right).
\end{aligned}$$

Using (6.5) again, find the real part of the integral in the previous expression. We finally obtain the following asymptotics in time for the concentration $h(z,t)$:

$$\begin{aligned}
h(z,t) &= c + A\cos\alpha e^{mz\sin^2\alpha}\sin\left(\omega t + \frac{mz}{2}\sin 2\alpha - \alpha\right) + O(e^{-kt}),\\
\frac{\partial h}{\partial t} &= A\omega\cos\alpha e^{mz\sin^2\alpha}\cos\left(\omega t + \frac{mz}{2}\sin 2\alpha - \alpha\right) + O(e^{-kt}),
\end{aligned} \tag{6.6}$$

where $\sin\alpha = \omega/\sqrt{\omega^2+k^2}$ and $\cos\alpha = k/\sqrt{\omega^2+k^2}$.

Note that this asymptotic formula can be differentiated with respect to z and t and these derivatives satisfy analogous asymptotic formulas. Therefore, from the equations and boundary conditions (1.1) and (1.2) we find the asymptotic formulas for the liquid and gas velocities:

$$v_L(z,t) = -A\omega\cos\alpha \int\limits_0^z e^{m\xi\sin^2\alpha}\cos\left(\omega t + \frac{m\xi}{2}\sin 2\alpha - \alpha\right)d\xi + \varphi(t) + O(e^{-kt}), \tag{6.7}$$

$$v_G(z,t) = A\omega\cos\alpha \int\limits_0^z e^{m\xi\sin^2\alpha}\cos\left(\omega t + \frac{m\xi}{2}\sin 2\alpha - \alpha\right)d\xi + \psi(t) + O(e^{-kt}). \tag{6.8}$$

The mean concentration over the period $\tau_0 = 2\pi/\omega$ is asymptotically equal to

$$\bar{h}(z,t) = \frac{1}{\tau_0}\int\limits_t^{t+\tau_0} h(z,\theta)\,d\theta = c + O(e^{-kt}).$$

Thus,

(1) if $F(t)$ is a periodic function then h, v_L, v_G, and $\bar{h}(t)$ tend to periodic functions with the same period as time tends to infinity;

(2) if the boundary data $\psi(t)$ and $\varphi(t)$ tend to constants and $\lambda \le \gamma$ then h, v_L, v_G, and $\bar{h}(t)$ tend to constants as t tends to infinity.

The analysis of the resulting asymptotic formula for solutions to the problem shows that the amplitude of liquid concentration grows with respect to the coordinate z exponentially, the waves of high concentration liquid "run" along the column with the same frequency as small perturbations of the input velocities of the liquid and/or gas at the packed-bed column inlet. Moreover, there is a shift in the oscillation phase of the waves depending on the spatial coordinate, which also well describes the observed experimental facts. For describing not only a qualitative picture of the experiment but also a quantitative comparison with the experiment, in [2], there were chosen experimentally varied parameters of the model and the column characteristics for various kinds of liquid and gas. Then, using the constructed model with the so-found parameters, the possibility was considered of the theoretical minimization of the energy losses averaged over the column length and the period of the sinusoidal oscillations of the velocity at the inlet. For different values of the varied parameters, in comparing with the stationary input velocities, it turned out that all cases are realized for the model (see [9]):

(a) the mean energy losses cannot be reduced in principle by periodic oscillations of the inlet gas and liquid velocities for any oscillation amplitudes of phase velocities;

(b) the mean energy losses can always be reduced by periodic oscillations of the inlet gas and liquid velocities for any oscillation amplitudes of the inflows of phase velocities;

(c) the mean energy losses can be reduced conditionally by periodic oscillations of the inlet gas and liquid velocities under a special synchronization of the oscillation amplitudes of the inflows of phase velocities.

The possibility of occurrence of one of the cases (a)–(c) depends (as was established theoretically for the model), in particular, on the regulated values of the parameters: the oscillation frequency ω and the column length L. The model theoretically predicts the "mathematical" fact (not yet explained by the chemical process engineers) that cases (a)–(c) may be realized regardless of the parameters ω and L.

Despite a sufficient simplicity of the model, it not only describes the waves running with an increase of their amplitudes and observed in the cocurrent column, but also predicts a theoretical possibility of overshoot phenomena in a countercurrent column when the inlet velocities of the gas and liquid phases are given at the opposite ends of the column and the kinetic parameter k tends to infinity [10].

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